Web Appendix For "Vertical integration, exclusive dealing, and ex post cartelization" by Yongmin Chen and Michael Riordan

The Web Appendix extends our analysis to two models of multiple downstream firms.

Hub and spokes model

We first develop a new model of price competition by multiple downstream firms that is a natural extension of the duopoly model. In addition to extending our results, the model may also have independent interest in suggesting a new way of modeling non-localized price competition by differentiated oligopolists. To save space, we shall make our arguments mostly informally; and, we continue to assume that contracts are bilateral and public.

Suppose that the downstream has $n \geq 2$ firms, *D1,D2,...Dn.* As before, *D1* and *U1* are vertically integrated. Each Di is associated with a line of length $\frac{1}{2}$, l_i. The two ends of l_i are called origins and terminals, respectively. Firm Di is located at the origin of l_i, and the lines are so arranged that all the terminals meet at one point, the center. This forms a network of lines connecting competing firms ("spokes"), and a firm can supply the consumer only by traveling on the lines. *Ex ante*, the consumer is located at any point of this network with equal probabilities. The realized location of the consumer is fully characterized by a vector (l_i, x_i), which means that the consumer is on l_i with distances of x_i to Di and of $\frac{1}{2} - x_i + \frac{1}{2} = 1 - x_i$ to *Dj*, $j \neq i$.[1] Obviously, the linear duopoly model is a special case of the hub-and-spokes model with $n = 2$.

As in our earlier analysis, consider first the case where $U1$ is a monopolist in the upstream market. A contract offered by *U1* to Dj, $j = 2, ...n$, can be written as (t_j, r_j). Modifying equations (1) and (2), we can define $P_1^m(x_1)$ and $P_j^m(x_j, r_j)$ as

[1] For the consumer located at the center, we shall denote her by $\left(l_1, \frac{1}{2}\right)$.

satisfying

$$\begin{aligned} P_1^m(x_1) - c - \tau x_1 &= \frac{1 - F\left(P_1^m(x_1)\right)}{f\left(P_1^m(x_1)\right)}, \\ P_j^m(x_j, r_j) - r_j - \tau x_j &= \frac{1 - F\left(P_j^m(x_j, r_j)\right)}{f\left(P_j^m(x_j, r_j)\right)}, \; j = 2, ..., n. \end{aligned}$$

Let $\bar{r} \equiv \min\{r_j : j = 2, ..., n\}$. Modifying equations (3) and (4) in Section 3, for $i = 1, ..., n$ and $j = 2, ..., n$, we can define

$$P_1((l_i, x_i), \bar{r}) = \begin{cases} \min\left\{P_1^m(x_1), \bar{r} + \tau(1 - x_1)\right\} & \text{if } i = 1 \\ \min\{P_1^m\left(1 - x_i\right), \bar{r} + \tau(1 - x_i)\} & \text{if } i \neq 1 \end{cases},$$

$$P_j((l_i, x_i), r_j, \bar{r}) = \begin{cases} \min\left\{P_j^m(x_j, r_j), \max\{r_j + \tau x_j, \min\{P_1^m(1 - x_j), \bar{r} + \tau(1 - x_j)\}\}\right\} & \text{if } i = j \\ r_j + \tau(1 - x_i) & \text{if } i \neq j \end{cases}.$$

Extending Lemma 1, in any downstream pricing game following any given $\{(t_j, r_j) : j = 2, ..., n\}$, there is a unique (refined) equilibrium outcome,[2] in which *D1* sets $P_1((l_i, x_i), \bar{r})$ and *Dj* sets $P_j((l_i, x_i), r_j, \bar{r})$, with the equilibrium price for consumer (l_i, x_i) being

$$\begin{aligned} & P^*((l_i, x_i), r_i, \bar{r}) \\ = & \begin{cases} \min\left\{P_1^m(x_1), \bar{r} + \tau(1 - x_1)\right\} & if \quad i = 1 \\ \min\left\{P_i^m(x_i, r_i), \max\{r_i + \tau x_i, \min\{P_1^m\left(1 - x_i\right), \bar{r} + \tau(1 - x_i)\}\right\} & if \quad i \neq 1 \end{cases}; \end{aligned}$$

consumer (l_i, x_i) selects *D1* if $i = 1$ or if $i \neq 1$ but $\min\{P_1^m\left(1 - x_i\right), \bar{r} + \tau(1 - x_i)\} < r_i + \tau x_i$; and consumer (l_i, x_i) selects *Di* if $i \neq 1$ and $\min\{P_1^m\left(1 - x_i\right), \bar{r} + \tau(1 - x_i)\} \geq r_i + \tau x_i$. As in Lemma 1, $P_1^m\left(\frac{1}{2}\right) \geq r_i + \frac{1}{2}\tau$ for any equilibrium contract (t_i, r_i).

The presence of additional downstream firms introduces several issues that we must consider in extending the analysis leading to Proposition 1.

[2] As in standard Bertand competition with more than two firms, the strategy profile supporting the unique equilibrium outcome may not be unique.

First, it is now possible that $r_j \neq r_k$ for some $j, k = 2, ..., n$ and $j \neq k$. Suppose that $r_k = \bar{r} < r_j$ for some $j = 2, ..., n$; i.e., *Dk* has a cost advantage in supplying (l_j, x_j) when $r_k + \tau(1 - x_j) < r_j + \tau x_j$. But *Dk* cannot benefit from selling to such a consumer, since the competition from *D1* will drive the price down to $\min\{P_1^m(1 - x_j), r_k + \tau(1 - x_j)\} \leq r_k + \tau(1 - x_j)$. This is because the perceived marginal cost for *D1* in supplying such a consumer when *Dk* is the other potential supplier and purchases from *U1* at r_k, is $c + r_k - c = r_k$.

Second, it immediately follows that to maximize joint upstream-downstream industry profits, we must have $(t_j, r_j) = (t, r)$ for $j = 2, ..., n$; because, if $r_k < r_j$ for some $j \neq k$, then slightly lowering r_j has no effect on the competition for consumer (l_i, x_i), $i \neq j$ but increases the expected industry profit from consumer (l_j, x_j). This allows us to generalize equations (5) and (6) and define

$$\begin{aligned}\Pi(r) &= \frac{2}{n}\int_0^{\frac{1}{2}} \left[P_1(x, r) - \tau x - c\right]\left[1 - F\left(P_1(x, r)\right)\right] dx + \\ &\quad \frac{n-1}{n} 2\int_0^{\frac{1}{2}} \left[P_2(x, r) - \tau x - c\right]\left[1 - F\left(P_2(x, r)\right)\right] dx,\end{aligned}$$

$$t(r) = \frac{2}{n}\int_0^{\frac{1}{2}} \left[P_2(x, r) - \tau x - r\right]\left[1 - F\left(P_2(x, r)\right)\right] dx,$$

where $\Pi(r)$ is the joint industry profits when $(t_j, r_j) = (t(r), r)$ for all $j = 2, ...n$. The transfer $t(r)$ fully extracts rents from the downstream industry.

Notice that an increase in r has the similar trade off here as in the downstream duopoly case: it affects positively the profit for *D1* due to relaxed competition, but affects negatively the profits for each *Dj* if it worsens the double mark-up distortion. Since the second effect is more important with a higher n, we conclude that $\hat{r}$ decreases in n, where

$$\hat{r} = \arg\max_{c \leq r \leq \bar{v}} \{\Pi(r)\}.$$

As in Proposition 1, we will have $c \leq P_1^m(0) - \tau < \hat{r} < P_1^m\left(\frac{1}{2}\right) - \frac{1}{2}\tau$, and define $\hat{t} = t(\hat{r})$.

We can thus extend Proposition 1 to the hub-and-spokes model with $n \geq 2$ downstream competitors.

Proposition 1a The game where U1 is the only upstream supplier has a unique equilibrium, in which U1 offers Dj contract $\left(\hat{t}, \hat{r}\right)$, which is accepted by Dj, $j = 2, ..., n$. Di is the potential seller with price $P^*((l_i, x_i), \hat{r}, \hat{r})$ if the consumer is located at (l_i, x_i), $i = 1, ..., n$. Furthermore, $c \leq P_1^m(0) - \tau < \hat{r} < P_1^m\left(\frac{1}{2}\right) - \frac{1}{2}\tau$, and $\hat{r}$ decreases in n.

Thus, just as in the downstream duopoly model, the firm that is nearest to the consumer will bid the lowest price and will make the sale if this price does not exceed the consumer's valuation. The equilibrium $\hat{r}$ is above c for the same reason as in the duopoly case: it reduces downstream competition and thus raises industry profits.

Returning to upstream duopoly, when *Dj* contracts to purchase from *U2* at $(0, c)$, *D1* will charge $c + \tau(1 - x_j) < \min\{P_1^m(1 - x_j), \hat{r} + \tau(1 - x_j)\}$ if the consumer is located at (l_j, x_j) and $j \neq 1$, and thus the (expected) joint profit of *U2-Dj* is

$$\frac{2}{n}\int_0^{\frac{1}{2}} \tau(1-2x)\left[1 - F\left(c + \tau(1-x)\right)\right] dx,$$

which is lower than the joint *U1-Dj* profit under $\hat{r}$.

Since the expected profit of $D2$ when it contracts with $U1$, excluding any transfer payment, is

$$\begin{aligned} & \frac{2}{n}\int_0^{\frac{1}{2}} \left[P^*((l_2, x), \hat{r}, \hat{r}) - \tau x - \hat{r}\right]\left[1 - F\left(P^*((l_2, x), \hat{r}, \hat{r})\right)\right] dx \\ = & \frac{2}{n}\int_0^{\frac{1}{2}} \left[P_2(x, \hat{r}) - \tau x - \hat{r}\right]\left[1 - F\left(P_2(x, \hat{r})\right)\right] dx, \end{aligned}$$

we can modify equation (7) to define

$$t^* = \frac{2}{n}\int_0^{\frac{1}{2}} \left[P_2(x, \hat{r}) - \tau x - \hat{r}\right]\left[1 - F\left(P_2(x, \hat{r})\right)\right] dx - \frac{2}{n}\int_0^{\frac{1}{2}} \tau(1-2x)\left[1 - F\left(c + \tau(1-x)\right)\right] dx,$$

where $t^* < 0$.

We can thus extend Proposition 2 as follows:

Proposition 2a The game where the upstream market is a duopoly has an equilibrium in which U2 offers Dj contract $(0, c)$ and U1 offers Dj contract $(t^*, \hat{r})$, and Dj contracts with U1, $j = 2, ..., n$. This downstream outcome is the same as under upstream monopoly.

The intuition here is the same as in the downstream duopoly case: When the integrated firm supplies *D2, ..., Dn* at a price above marginal cost, the former has less incentive to undercut the latter because of the opportunity cost of foregone input sales to *Dj*. This dampening of horizontal competition explains *U1*'s advantage and ability to preempt *U2.* The r that is optimal under upstream monopoly is again chosen to maximize the joint industry profits, and t^* is chosen so that each stand-alone firm is willing to enter the exclusive contract with *U1.* If any $Dj, j = 2, ..., n$, deviates and contracts with *U2* at $(0, c)$, *D1* will reduce its price to $c + \tau(1 - x_i)$ for any consumer located at (l_i, x_i), $i \neq 1$, making the expected joint profit between *U2-Dj* lower than the expected joint profit between *U1-Dj* under $\hat{r}$, which implies that no deviation would occur.[3]

Since $\hat{r} > c$, just as in the downstream duopoly case, the use of exclusive contracts is crucial for *U1* to be able to exclude *U2* and to raise the downstream prices.

We can further show that, as in the downstream duopoly case, if *U1* and *D1* are vertically separated, there exists an equilibrium in which exclusive contracts are irrelevant due to competitive (marginal cost) contracting for the intermediate good. For purposes of this discussion we suppose there are m equally efficient upstream firms, indexed $i = 1, ...m$, and n downstream firms, indexed $j = 1, ...n$. Proposition 2' extends readily to this generalization, with Ui, $i = 3, ...m$ acting the same as *U2.* The discussion below pertains to vertically-separated oligopolies.

First, we can argue that it is an equilibrium for all *Ui* to offer $(0, c)$ to all down-

[3]Notice that since in equilibrium *U2* offers $(0, c)$, adding additional upstream firms that are the same as *U2* will not change the results.

stream firms and *U1*'s offer is accepted by all *Dj*, $j = 1, ..., n$. It suffices to consider deviations by an upstream firm, say *U2,* that offer any (t, r), $r > c$, to all *Dj*, $j = 1, ..., n$.[4] For *Dj* to be willing to accept the deviation contract, it is necessary that *Di* receives a payment that compensates it for the loss in profit due to $r > c$, or

$$\begin{aligned} -t &\geq \frac{2}{n}\int_0^{\frac{1}{2}} \tau(1-2x_i)\left[1-F\left(c+\tau(1-x_i)\right)\right]dx_i \\ &\quad -\frac{2}{n}\int_0^{\max\left\{0,\frac{1}{2}-\frac{r-c}{2\tau}\right\}} \left[\left(c+\tau\left(1-x_i\right)\right)-\left(r+\tau x_i\right)\right]\left[1-F\left(c+\tau(1-x_i)\right)\right]dx_i \\ &> \frac{2}{n}\int_0^{\frac{1}{2}} (r-c)\left[1-F\left(c+\tau(1-x_i)\right)\right]dx_i. \end{aligned}$$

With the deviation, *U2*'s revenue from *Dj* is

$$(r-c)\frac{2}{n}\int_0^{\frac{1}{2}} \left[1-F\left(\min\{P_2^m\left(x_i,r\right), r+\tau(1-x_i)\}\right)\right]dx_i.$$

Therefore, with n firms,

$$\begin{aligned} & n\left[(r-c)\frac{2}{n}\int_0^{\frac{1}{2}} \left[1-F\left(\min\{P_2^m\left(x_i,r\right), r+\tau(1-x_i)\}\right)\right]dx_i\right] - n\left(-t\right) \\ < \;& (r-c)\,2\left[\int_0^{\frac{1}{2}} \left[1-F\left(c+\tau(1-x_i)\right)\right]dx_i - \int_0^{\frac{1}{2}} (r-c)\left[1-F\left(c+\tau(1-x_i)\right)\right]dx_i\right] \\ = \;& 0, \end{aligned}$$

which implies that there can be no profitable deviation from the candidate equilibrium.

Second, we can rule out equilibria in which any downstream firm, say *Dj*, contracts to purchase at $r_j > c$ with the additional parameter restriction $m \geq \frac{n}{2} + 1$. Recall that the basic intuition for this result under downstream duopoly is the following: if *U1* contracts with *D1* and *U2* contracts with *D2,* then each pair would maximize its

[4] A deviation aimed at a strict subset of downstream firms is even less profitable, because the upstream firm must compensate for sales lost to the remaining downstream competitors for whom $r_i = c$.

joint profit by setting the price from *Ui* to *Di* at c; if one of the upstream firms, say *U1*, contracts with both *D1* and *D2* at some price above c, *U2* can offer contracts to either *D1* or *D2* with price c and achieve a higher joint profit with either of them than the joint profit between either *U1-D1* or *U1-D2.* This intuition extends to multiple downstream competitors. Since $m \geq \frac{n}{2} + 1$, at least one upstream firm, say *U2*, is either not contracting with any downstream firm or is contracting with only one downstream firm at any possible equilibrium. If *U2* is contracting with only one downstream firm, say *D2* at r_2, then it is optimal for $r_2 = c$, which implies that it would not be optimal for any other pair of upstream and downstream firms to have the intermediate good price above c. If *U2* is not contracting with any downstream firm and some upstream firm, say *U1*, is contracting with one or several *Dj* with $r_j > c$, then *U2* can offer $r_j = c$ to one of the *Dj,* which will be accepted. (It is important to see why the cartelization equilibrium can be sustained under *U1-D1* integration but not under vertical separation. Under *U1-D1* integration, a deviation to $r = c$ for some *Dj* would be met with a reduction of *D1*'s perceived marginal cost from $\hat{r}$ to c when *D1* competes with *Dj*, which makes the deviation unprofitable; while under *U1-D1* separation, the other downstream's marginal costs are taken as given when *Dj* considers deviation.)

Notice that competitive contracting is still an equilibrium under vertical separation even without the parameter restriction on m. Thus this restriction is not crucial for our main insight about the effect of vertical integration. However, for $m < \frac{n}{2} + 1$, we have not ruled out the possibility of another vertical-separation equilibrium in which each upstream firm contracts with several downstream firms at some price above c. In such a situation, an under-cutting upstream firm must balance the considerations of its own part of the downstream market and the rest of the downstream market. This complication is avoided with the assumption $m \geq \frac{n}{2} + 1$.[5]

[5] The restriction is unnecessary if contracts are private (Chen and Riordan, 2003). The same is

Circle model

We next consider an alternative way of extending our model to multiple downstream firms. Instead of considering non-localized competition in the downstream market, we consider localized competition, adopting the circular city model of Salop (1979). Assume that the consumer is located with equal chance at any point of a circle with a perimeter equal to 1. Firms are located equidistant from each other on the circle. With $n > 2$ firms, *D1,D2,...,Dn,* the distance between any two neighboring firms is simply $\frac{1}{n}$. Let *D1* be located at the bottom of the circle, followed clockwise by *D2, ..., Dn*. Thus, *D1*'s neighboring firms on the left and on the right are denoted as *D2* and *Dn*, respectively. The realized location of the consumer is denoted as $x \in [0,1]$, where $x = 0$ if the consumer is at the bottom of the circle (the position of *D1*), and x increases clockwise (so, for instance, $x = \frac{1}{2}$ if the consumer is located at the top point of the circle). In what follows we shall only sketch our analysis, under the same contracting assumptions as in the hub-and-spokes model and assume $m \geq \frac{n}{2} + 1$. As in the hub-and-spokes model, the parameter restriction rules out the possibility of additional equilibria.

If *U1* and *D1* are vertically separated, then again the only equilibrium outcome is for all downstream firms to purchase the input at price c, same as in our basic model with rather similar reasoning.[6] In what follows we thus assume that *U1* and *D1* are vertically integrated. For convenience, we shall focus on the case $n = 4$, and will in the end discuss the cases $n > 4$ and $n = 3$.

With $n = 4$, *D1* competes with *D2* and *D4* respectively when $x \in [0, \frac{1}{4}]$ and $x \in [\frac{3}{4}, 1]$, *D2* competes with *D3* when $x \in [\frac{1}{4}, \frac{1}{2}]$, and *D3* competes with *D4* when $x \in [\frac{1}{2}, \frac{3}{4}]$. Notice that the only firm *D1* does not compete with directly is *D3*. Denote

true for the circle model that follows.

[6] Without the restriction $m \geq \frac{n}{2} + 1$, this equilibrium outcome is still valid, but we do not rule out other possible equilibria. The restriction avoids the complication.

the contract *Dj* accepts by (t_j, r_j), $j = 2, 3, 4$.

As before, if *U1* were the only upstream producer, then in equilibrium $(t_j, r_j) = (t^*, r^*)$ and $r^* > c$, and we can extend Proposition 1 to the following:

The game where U1 is the only upstream supplier has a unique equilibrium. At this equilibrium, $r_i^* = r^*$, $i = 2, 3, 4$, for some $r^* > c$. D1 is the potential seller when $x \in [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$, D2 is the potential seller when $x \in [\frac{1}{8}, \frac{3}{8}]$, D3 is the potential seller when $x \in [\frac{3}{8}, \frac{5}{8}]$, and D4 is the potential seller when $x \in [\frac{5}{8}, \frac{7}{8}]$.

More interesting is what happens under upstream competition ($m \geq 2$), to which we now return. We sketch our argument in two parts:

(1) $r_3^* = c$ in equilibrium.

Because *U1* is integrated with *D1,* and *D1* competes directly with *D2* and *D4*, *U1* has an advantage over *Ui,* $i = 2, ..., m$, in achieving any potential downstream collusive outcome. It thus suffices to argue that if in equilibrium *U1* contracts with all three independent downstream firms, *Dj*, $j = 2, 3, 4$, we must have $r_3^* = c$. To make this argument, we notice that r_3 only affects the competition for $x \in [\frac{1}{4}, \frac{3}{4}]$, or the top half of the circle. In equilibrium, we must have $r_2^* = r_4^*$, and due to symmetry we can focus on the segment $x \in [\frac{1}{4}, \frac{1}{2}]$ and consider profits on that segment. For any given r_2^*, if $r_3^* > c$, *Ui* could offer a contract to *D3* at $r_3 = c$ that maximizes their joint profit, and this profit, same as the joint profit of *U1-D3* if they contract under $r_3 = c$, is higher than the joint profit of *U1-D3* with $r_3^* > c$. Furthermore, an offer from *Ui* to *D2* with $r_2 = c$ would enable *Ui-D2* to earn a higher joint profit when $r_3^* > c$ than when $r_3^* = c$. Therefore, to prevent *D2* and *D3* to accept a competitive contracting offer from *Ui*, it costs *U1* more under $r_3^* > c$ than under $r^* = c$. Thus, it is optimal for *U1* to contract with *D3* at $r_3^* = c$. Notice that if it is an equilibrium for *U1* to contract with *D3* at $r_3^* = c$, it is also an equilibrium for *Ui* to contract with *D3* at $r_3^* = c$.

(2) In equilibrium, *U1* is able to raise the input price of its neighbors; i.e., $r_2^* > c$ and $r_4^* > c$, and to raise the final price for the consumer.

We shall look for r_2 and r_4 such that the joint profits of *U1-D1-D2* are maximized when the consumer is located on the left half of the circle and the joint profits of *U1-D1-D4* are maximized when the consumer is located on the right half of the circle. (Note that we already know $r_3^* = c$.) Because of symmetry, the equilibrium r_2^* and r_4^* would be equal.

For consumer x located between *D1* and *D2* ($x \in [0, \frac{1}{4}]$), the consumer's distances from *D1* and *D2* are x and $\frac{1}{4} - x$, respectively. Since the distance of consumer x from *D3* is $\frac{1}{2} - x$, in order for the consumer to be served by either *D1* or *D2*, we need

$$r_2 + \left(\frac{1}{4} - x\right)\tau \leq c + \left(\frac{1}{2} - x\right)\tau,$$

or[7] $r_2 \leq c + \frac{1}{4}\tau$. But since $c + \frac{1}{4}\tau < c + \tau \leq P_1^m(0)$, it follows that, for any $x \in [0, \frac{1}{4}]$, in equilibrium *D1* and *D2* will charge prices that are below their unconstrained monopoly prices. The equilibrium prices for consumer x are thus equal to $\max\{r_2 + \tau(\frac{1}{4} - x), r_2 + \tau x\}$, and *D1* and *D2* each serves the consumer located between $[0, \frac{1}{8}]$ and $[\frac{1}{8}, \frac{1}{4}]$, respectively.

For consumer $x \in [\frac{1}{4}, \frac{1}{2}]$, for whom *D2* and *D3* compete, the marginal consumer is $\hat{x}_2 = \frac{c - r_2}{2\tau} + \frac{3}{8}$, where *D2* serves if $x \in [\frac{1}{4}, \hat{x}_2]$ with price $c + (\frac{1}{2} - x)\tau$ and *D3* serves if $x \in [\hat{x}_2, \frac{1}{2}]$.

Therefore, the expected joint profit of *U1-D1-D2* when the consumer is located on the left half of the circle is

$$\begin{aligned}\Pi(r_2) &= 2\int_0^{\frac{1}{8}} \left[r_2 + (\frac{1}{4} - x)\tau - (c + x\tau)\right]\left[1 - F\left(r_2 + (\frac{1}{4} - x)\tau\right)\right] dx \\ &\quad + \int_{\frac{1}{4}}^{\hat{x}_2} \left[c + (\frac{1}{2} - x)\tau - (c + (x - \frac{1}{4})\tau)\right]\left[1 - F\left(c + (\frac{1}{2} - x)\tau\right)\right] dx.\end{aligned}$$

[7] If this condition is not satisfied, then $D3$ would compete with $D1$ for consumer $x \in [0, \frac{1}{4}]$. By lowering r_2 to $c + \frac{1}{4}\tau$, the price for x is not changed but the profits to $D3$ would go to $D2$. Thus, to look for the optimal r_2, we need to restrict to $r_2 \leq c + \frac{1}{4}\tau$.

Let

$$\hat{r}_2 \equiv \arg \max_{c \leq r_2 \leq c+\frac{1}{4}\tau.} \Pi(r_2).$$

Then, since

$$2\int_0^{\frac{1}{8}} \left[r_2 - c + (\frac{1}{4} - 2x)\tau\right] \left[1 - F\left(r_2 + (\frac{1}{4} - x)\tau\right)\right] dx$$

is strictly increasing in r_2 at $r_2 = c$, while

$$\left.\frac{d\left[\int_{\frac{1}{4}}^{\hat{x}_2} (\frac{3}{4} - 2x)\tau \left[1 - F\left(c + (\frac{1}{2} - x)\tau\right)\right] dx\right]}{dr_2}\right|_{r_2=c}$$

$$= \left.(\frac{3}{4} - 2\hat{x}_2)\tau \left[1 - F\left(c + (\frac{1}{2} - \hat{x}_2)\tau\right)\right] \left(-\frac{1}{2\tau}\right)\right|_{r_2=c} = 0,$$

we must have $\Pi'(r_2)|_{r_2=c} > 0$, and thus $\hat{r}_2 > c$.

If *D2* were to contract with *Uj*, the contract that would maximize the joint profit of *Ui-D2* and give all this profit to *D2* is $(0, c)$. The joint profit of *U1-D1-D2* when the consumer is located on the left half of the circle would then be $\Pi(c) < \Pi(\hat{r}_2)$. Notice that $D2's$ profit when it accepts $(0, c)$ from *U2* is $\frac{2}{3}\Pi(c)$, and *U1-D1*'s profit from this part of the circle is $\frac{1}{3}\Pi(c)$.

Now let t_2^* be such that $D2's$ profit when it accepts $(t_2^*, \hat{r}_2)$ from *U1* is $\frac{2}{3}\Pi(c)$. Then, $D2's$ profit when it accepts $(t_2^*, \hat{r}_2)$ from $U1$ is the same as that when it accepts $(0, c)$ from Uj, and *U1* will indeed offer $(t_2^*, \hat{r}_2)$ to *D2* since $\Pi(\hat{r}_2) - \frac{2}{3}\Pi(c) > \frac{1}{3}\Pi(c)$. Therefore, corresponding to Proposition 2, we have:

The game where the upstream market has $m \geq 2$ equally efficient firms has a unique equilibrium outcome, where U1 contracts with D2 and D4 at $(t_2^*, \hat{r}_2)$, while D3 contracts with either U1 or Uj, j $\neq 1$, at $(0, c)$.

Importantly, however, now the downstream equilibrium outcome is *different* from under upstream monopoly. The vertically integrated firm is able to raise input prices only for its neighbors, using exclusive contracts. If *U1-D1* attempts to contract with the non-neighboring firm, *D3,* at $r_3 > c$, and if *D3* instead accepts *U2*'s offer at c, *U1-D1* cannot "punish" *D3* with a reduction of *D1*'s opportunity cost from r_3 to c, since *D1* does not compete directly with *D3* and r_2^* is given. Localized downstream competition thus reduces the vertically integrated firm's ability to cartelize the downstream market. More generally, if $n > 4$, in equilibrium we have $r_2^* = r_n^* > c$ and $r_j^* = c$ for $j = 3, ..., n-1$.[8]

The $n = 3$ case is different because *D2* and *D3* compete directly both with *U1* and with each other. Consequently the joint profit of *U1-D1-D2* depends on r_3. By the theorem of the maximum there exists a continuous bounded function $\sigma(r_3)$ such that $r_2 = \sigma(r_3) \geq c$ maximizes the joint profit of *U1-D1-D2* given any $r_3 \geq c$, and by Brouwer's theorem there exists a fixed point $r^* = r_2(r^*)$ that defines a symmetric equilibrium $r_3^* = r_2^* = r^*$. Finally, the joint profit of *U1-D1-D2* is increasing in r_2 when $r_2 = c$, which implies $r^* > c$.

Therefore, our main results also hold in the circle model of downstream competition: vertical integration in combination with exclusive contracts excludes an equally efficient supplier and partially cartelizes the downstream industry; neither of these practices alone can be counted to achieve these effects. However, the extent of upstream foreclosure and downstream cartelization depends on the nature of competition—whether it is localized or non-localized, in addition to on the level of concentration in the downstream market.

[8] The restriction $m \geq \frac{n}{2}+1$ rules out the possibility of other equilibria that involve non-competitive contracting for firms beyond *D1*'s two neighbors.